# Event Based Control of Stochastic Linear Systems

Dipankar Maity and John S. Baras
Department of Electrical and Computer Engineering, Institute for Systems Research
University of Maryland College Park, Maryland, USA 20742
Email: {dmaity, baras}@umd.edu

***Abstract*—In this paper, an event based control strategy for linear systems under stochastic disturbances is presented where the control input generator generates a control which tries to mimic a state feedback control between two successive events. The control input is generated in such a way that the error between the state of this system and the continuous state feedback system is bounded. The event generator triggers an event based on this error and the control input generator modifies its control input in such a way that it corroborates that the error generated up to this triggering instance is reduced to zero and thus it does not have any effect on the future states regardless of the stability of the plant. It is also shown that under this event triggering mechanism, the control input generator generates the control in such a way that the error bound could be made arbitrarily small.**

## I. Introduction

Control of a large system generally requires continuously reading the sensor measurements, transmitting it to the control input generators and computing the control law. Control, communication and computing are integrated in an inseparable way. In a centralized system, the performance depends on the continuity of communication and computing the control law accurately. In a distributed system, although the controller is implemented distributively, however, it also requires continuous interaction among the subsystems. Communication and data transmission are an indispensable part for networked control systems. As a result, their performance is generally determined by the available network bandwidth and computing resources. Scarcity of communication bandwidth for continuous communication is a limitation for these systems.

In the recent past, researchers have proposed novel techniques that require only discrete-time communication to overcome the problem of bandwidth limitation. Event-Based control [1], self triggered control [2], [3] and periodic time control [4], [5], [6] are some efficient ways to reduce the communication overhead. The essence behind these techniques is to communicate at discrete time instances rather than communicating continuously. In periodic time control, the communication is done periodically after $T$ amount of time. The main challenge in this approach is to find suitable time period $T$ to guarantee acceptable performance. In self triggered and event based control, the communication is done only when some event has occurred. Event based control monitors the plant measurements and it triggers for communication based on some signal measurements. On the other hand, self triggered control only measures the state of the system and triggers when the state deviates by some threshold from its value at the last triggering instance.

Event based control has been proved to be very effective and hence it attracted a great deal of research in the last few decades. In [7], a comparison has been made between the performance of event based control and periodic control. It has been shown there that under some conditions the event based control performs better than periodic control. A simple PID controller is proposed in [8] for event based control which reduces large CPU computation at the cost of minor control performance degradation. In the literature, asynchronous control [9], event based sampling [10], event driven sampling [11], Lebesgue sampling [1], deadband sampling [12] have been proposed to carry out the idea of supplicating communication only when some event has been occurred. In [13], the authors analyzed event based control in a stochastic setting. In [14], [15], [16], a different approach is taken to reduce the information content rather than reducing the communication frequency. Recently, a state feedback approach for an event based system is considered in [17] where the feedback control is generated from another system which is updated every time a trigger is introduced. Event based control for distributed interconnected linear systems is proposed in [18] and [19]. A quadratic approximated value function is used in [20] to perform event based control.

In event based control, self triggered control or periodic control, the controller being unable to access the continuous state, approximates the state and the approximated state is used to produce the control input. Since the generated control input is different from the actual feedback control, the response of the system is not as it would have been if there were a continuous state feedback. Hereafter, the state of the continuous state feedback system will be denoted as $x_c(t)$ and the state of the actual system as $x(t)$. The main purpose is to keep the error $e(t) = x(t) - x_c(t)$ as small as possible. Since the continuous state feedback is not used and the plant is affected by immeasurable noise, the reference signal $x_c(t)$ is not available for decision making. Apparently, it seems that no information is available about $e(t)$ and the triggering decision has to be taken based on some other signal value. In [17], the authors proposed another state variable which is used to generate the control input. This state variable, called dummy state ($x_d$) in our paper, was used to generate a signal, based on which the triggering decision was taken in [17]. In this paper, a new method will be proposed so that the knowledge of $e(t)$ can be obtained even though there is no information about $x_c(t)$. The proposed event triggering mechanism will generate events based on $e(t)$ and ensure satisfactory performance (Theorem IV.1). Moreover, in the existing approaches for event triggered control, no information about $e(t)$ is available even at the triggering instances and hence $e(t)$ cannot be controlled directly. In our approach, it will be shown that at each triggering instance the system is updated in such a way that will reduce the error $e(t_k)$

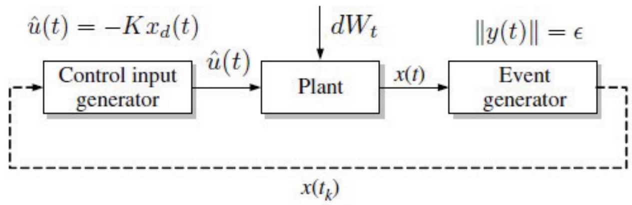


Fig. 1. Event based control loop with three subsystems: control input generator, the plant and the event generator. The communication link is shown in dashed line which allows communication only in discrete manner.

accumulated up to the $k$-th triggering instance.

The model for event based control and communication considered in this paper is similar to [17] and it is shown in Figure 1. However, unlike [17], no stability assumption are made for the plant dynamics (1) and the boundedness of the perturbations. The existing techniques consider time invariant cases [17], [18], [21], [22]; but general time varying systems will be considered in this paper. Additional remarks are made for time invariant systems (Theorems III.3, IV.3). It is clear that if the system is not asymptotically stable, the residual error, $e(t_k)$, after $k$-th triggering propagates up to the final time. For an unstable system it increases exponentially. The proposed approach assures that the control input can be designed (by introducing additional open loop control $\psi(t)$) in such a way that can mitigate this residual error.

The rest of the paper is organized as follows: Section II formulates the general problem that will be addressed; Section III provides the controller implementation, error analysis; Section IV describes the event triggering strategy, and optimal noise estimate for different disturbances; and Section V provides illustrative examples.

## II. Problem Formulation

Let us consider the linear stochastic dynamics of a system to be given by (1)

$$dx = Axdt + Budt + dW_t \tag{1}$$

where $x \in \mathbb{R}^n$ is the state of the system and $u$ is the control. $dW_t$ is an $n$ dimensional Wiener process acting as a noise to the system. Let the control be of the form $u = -Kx$ to achieve some system performance. The matrices $A, B$ and $K$ are time varying in general but the time dependencies will not be shown explicitly. Unless and otherwise stated, these matrices are assumed to be time varying. Moreover, no stability assumptions are made for these system matrices. In fact, in the simulations, examples will contain $A, B$ and $K$ such that $A - BK$ is not Hurwitz. Since the control input contains feedback term, it requires the state $x$ to be available continuously for all time. In a distributed system it requires continuous communication among the agents, which is not always possible due to network limitations.

The closed loop system with this control has the form (2)

$$dx_c = \tilde{A}x_c dt + dW_t \tag{2}$$

where $\tilde{A} = A - BK$. Since the communication is done in a discrete time manner, the exact state of the system, $x(t)$, is available only at those time instances, $t_i$. Since the state is not available continuously, the control $u$ has to be designed in a way such that it does not require the continuous state of the system and, nonetheless, it drives the new system to approximate the closed loop system (2) within the given tolerance level.

## III. Event Based Control: Closed Loop System and Error Analysis

This section describes the approach to calculate the control law without continuous information about the state $x$. An analysis for the error dynamics will be done in order to get some insight about how to calculate $e(t)$ when no information is available about $x_c(t)$.

### A. Control Input Generation

To start with this let us first find the actual control input for the closed loop system. From (2), it can easily be found that the state, $x_c(t)$, at time $t \geq t_k$, can be written as,

$$x_c(t) = \Phi_{\tilde{A}}(t, t_k)x_c(t_k) + \int_{t_k}^{t} \Phi_{\tilde{A}}(t, s)dW_s \tag{3}$$

where $\Phi_{\tilde{A}}(t, t_k)$ is the state transition matrix for $\tilde{A}$. Therefore, the control law at time $t \geq t_k$ can be given by equation (4).

$$u(t) = -K\Phi_{\tilde{A}}(t, t_k)x_c(t_k) - \int_{t_k}^{t} K\Phi_{\tilde{A}}(t, s)dW_s \tag{4}$$

If the noise $dW_s$ were not there, it is sufficient to know only $x(0)$ to calculate the control law for the entire time horizon. The presence of $dW_s$ makes it necessary to use $u = -Kx$ instead of (4). One suitable approach [17] will be to estimate $dW_t$ and use that information to calculate the control. Let a communication occur at time $t_k$ and for all $t \geq t_k$ the control input for (1) will take the form (5)

$$u_1(t) = -K\Phi_{\tilde{A}}(t, t_k)x(t_k) - \int_{t_k}^{t} K\Phi_{\tilde{A}}(t, s)\hat{W}_k ds \tag{5}$$

where $\hat{W}_k$ is the estimate of the process $W_t$ at time $t_k$. Therefore, $u_1(t) = u(t) + \int_{t_k}^{t} K\Phi_{\tilde{A}}(t, s)[dW_s - \hat{W}_k ds]$. The control law (5) has been proposed by [17]. As can be seen from the expression of $u$ and $u_1$, $u_1$ accumulates error with time and that affects the performance of $x$ as well. Every time the system communicates, it receives the value of $x$ at that time ($x(t_k)$ in the above case) and the control law $u_1(t_k)$ is equal to $u(t_k)$. However, there is no mechanism in this system to reduce the error incurred in $x$ due to the control input mismatch between two successive events. To overcome this fact the following control law is proposed:

$$\hat{u}(t) = u(t) + \int_{t_k}^{t} K\Phi_{\tilde{A}}(t, s)[\psi_k(s)ds - (\hat{W}_k ds - dW_s)] \tag{6}$$

The purpose of $\psi_k$ is to reduce the error due to the control input mismatch. Later in this paper, an explicit expression for $\psi_k$ is derived to guarantee desired performance. The control input (6) can be written as

$$\hat{u}(t) = -K[\Phi_{\tilde{A}}(t, t_k)x(t_k) + \int_{t_k}^{t} \Phi_{\tilde{A}}(t, s)[\hat{W}_k - \psi_k(s)]ds] \tag{7}$$

$\hat{u}(t)$ can be written as a state feedback control law whose dynamics are governed by the equation (8)

$$\begin{aligned} \dot{x}_d &= \tilde{A}x_d + \hat{W}_k - \psi_k(t) \qquad && x_d(t_k^+) = x(t_k) \\ \hat{u}(t) &= -Kx_d(t) && \forall t > t_k \end{aligned} \quad (8)$$

$t_k$ is the time when $k$-th event is triggered and $k$-th communication is done. After each communication, the dynamics of the dummy system (8) is changed and $x_d(t_k^+)$ is made equal to $x(t_k)$. (8) can be implemented independently without the knowledge of $x(t)$ and hence can be used to generate control $\hat{u}(t)$. The estimation of $\hat{W}_k$ can be done in a recursive fashion:

$$\hat{W}_k = \hat{W}_{k-1} + D^{-1}(x_d(t_k^+) - x_d(t_k)) \quad (9)$$

where $D = \int_{t_{k-1}}^{t_k} \Phi_{\tilde{A}}(t_k, s)ds$, and clearly $D^{-1}$ exists for all $t_k > t_{k-1}$ ($t_k$ and $t_{k-1}$ are the $k$-th and $k-1$-th triggering instances). This noise estimate was suggested by [17] and proved to be effective under certain situations.
With this control, the dynamics of the system (1) become

$$dx = Axdt - BKx_d dt + dW_t \quad (10)$$

### B. Error Dynamics

The dynamics of the event based closed loop system are given in (10). From (10) and (8) the dynamics of $x_e(t) = x(t) - x_d(t)$ are obtained as:

$$\begin{aligned} dx_e &= Ax_e dt + \psi_k(t)dt - \hat{W}_k dt + dW_t \\ x_e(t_k^+) &= 0 \qquad \forall t > t_k \end{aligned} \quad (11)$$

The dynamics of the error $e(t) = x(t) - x_c(t)$ is given by (12) with $e(0) = 0$

$$\dot{e} = \tilde{A}e + BKx_e \quad (12)$$

Let us define a new system, $\xi(t)$ whose dynamics are (13),

$$\begin{aligned} d\xi &= A\xi dt - \hat{W}_k dt + dW_t \\ \xi(t_k^+) &= 0 \qquad \forall t > t_k \end{aligned} \quad (13)$$

Defining $\phi(t) = \int_{t_k}^{t} \Phi_A(t, s)\psi_k(s)ds$ and using (11) and (13), (14) can be obtained.

$$x_e(t) = \xi(t) + \int_{t_k}^{t} \Phi_A(t, s)\psi_k(s)ds = \xi(t) + \phi(t) \quad (14)$$

Later on $\xi(t)$ will be used to trigger communications. Using the knowledge of $x$, $x_e$, $\xi$ and $\psi_k$, the aim is to control $e$ indirectly. Using (14) and (12) we get for all $t > t_k$,

$$e(t) = \eta(t) + \int_{t_k}^{t} \Phi_{\tilde{A}}(t, s)B(s)K(s)\xi(s)ds \quad (15)$$

where $\eta(t) = \Phi_{\tilde{A}}(t, t_k)e(t_k) + \int_{t_k}^{t} \Phi_{\tilde{A}}(t, s)BK\phi(s)ds$ and it satisfies the system (16)

$$\dot{\eta} = \tilde{A}\eta + BK\phi \qquad \eta(t_k) = e(t_k) \quad (16)$$

(15) relates $e$ with $\xi$ and (16) defines an independent system with control $\phi$. Although $e$ is not controllable in general, a part of $e$ (i.e. $\eta$) is directly controllable using $\phi$.

From the definition of $\phi(t)$ and (16), a new system $z = [\eta^T, \phi^T]^T$ is formed which follows the dynamics given in (17)

$$\dot{z} = Pz + Q\psi_k \qquad \forall t \geq t_k \qquad z(t_k) = [e(t_k)^T,\ 0^T]^T \quad (17)$$

where $P = \begin{pmatrix} \tilde{A} & BK \\ 0 & A \end{pmatrix}$ and $Q = [\mathbf{0}_{n\times n}, \mathbb{I}_{n\times n}]^T$.

The dynamical system (17) is very useful since it contains an independent control $\psi_k$ and the system can be stabilized (17) at the origin under certain conditions. Controlling $z$ to origin will make the error $e(t)$ to be $\int_{t_k}^{t} \Phi_{\tilde{A}}(t, s)B(s)K(s)\xi(s)ds$ i.e. the effect of the error accumulated up to time $t_k$, $e(t_k)$, on the future time will be nullified.

**Theorem III.1.** *If $\exists\delta > 0$, $\gamma \in \mathbb{R}^n$ such that $(R_{11} - R_{12}R_{22}^{-1}R_{21})\gamma = e(t_k)$, where $R_{ij} = \int_{t_k}^{t_k+\delta} S_i(t)S_j^T(t)dt$ $i, j \in \{1, 2\}$, $S_1(t) = \Phi_A(t_k, t) - \Phi_{\tilde{A}}(t_k, t)$ and $S_2(t) = \Phi_A(t_k, t)$, and the control law $\psi_k(t) = -[S_1(t) - R_{12}R_{22}^{-1}S_2(t)]^T\gamma$ for $t \in (t_k, t_k + \delta)$ and zero otherwise is applied, then $e(t) = \int_{t_k}^{t} \Phi_{\tilde{A}}(t, s)BK\xi(s)ds$ for all $t \geq t_k + \delta$.*

*Proof:* Considering the dynamics (17) we have

$$z(t) = \Phi_P(t, t_k)z(t_k) + \int_{t_k}^{t} \Phi_P(t, s)Q\psi_k(s)ds \quad (18)$$

It can be shown that $\Phi_p(s, r)$ has the following form (19)

$$\Phi_P(s, r) = \begin{bmatrix} \Phi_{\tilde{A}}(s, r) & \Phi_A(s, r) - \Phi_{\tilde{A}}(s, r) \\ 0 & \Phi_A(s, r) \end{bmatrix} \quad (19)$$

Therefore, $\Phi_P(s, r)Q = \begin{bmatrix} \Phi_A(s, r) - \Phi_{\tilde{A}}(s, r) \\ \Phi_A(s, r) \end{bmatrix}$.
For all $t \geq t_k + \delta$, from (18),

$$z(t) = \Phi_P(t, t_k)[z(t_k) + \int_{t_k}^{t_k+\delta} \Phi_P(t_k, s)Q\psi_k(s)ds]$$

From the expressions of $S_1$, $S_2$ and $\Phi_P(t_k, s)$, we get

$$z(t) = \Phi_P(t, t_k)[z(t_k) + \int_{t_k}^{t_k+\delta} \begin{bmatrix} S_1(s) \\ S_2(s) \end{bmatrix} \psi_k(s)ds]$$

Substituting the expression for the control $\psi_k(s)$,

$$\begin{aligned} z(t) =& \Phi_P(t, t_k)z(t_k) - \\ & \Phi_P(t, t_k) \int_{t_k}^{t_k+\delta} \begin{bmatrix} S_1(s) \\ S_2(s) \end{bmatrix} [S_1(s) - R_{12}R_{22}^{-1}S_2(s)]^T ds\gamma \\ =& \Phi_P(t, t_k)\Big(z(t_k) - \\ & \int_{t_k}^{t_k+\delta} \begin{bmatrix} S_1(s)S_1(s)^T - S_1(s)S_2(s)^T R_{22}^{-1}R_{12}^T \\ S_2(s)S_1(s)^T - S_2(s)S_2(s)^T R_{22}^{-1}R_{12}^T \end{bmatrix} \gamma ds\Big) \end{aligned}$$

Clearly from the expression of $R_{ij}$, $R_{ji} = R_{ij}^T$. Therefore,

$$z(t) = \Phi_P(t, t_k)\Big(z(t_k) - \quad \begin{bmatrix} R_{11} - R_{12}R_{22}^{-1}R_{21} \\ 0 \end{bmatrix} \gamma ds\Big)$$

Since $z(t_k) = [e(t_k)^t, 0^T]^T$ and $(R_{11} - R_{12}R_{22}^{-1}R_{21})\gamma = e(t_k)$, $z(t) = 0$ for all $t \geq t_k + \delta$ and this implies that $\eta(t) = 0$ for all $t \geq t_k + \delta$. Thus, from the expression of $e(t)$ in (15), $e(t) = \int_{t_k}^{t} \Phi_{\tilde{A}}(t, s)BK\xi(s)ds \quad \forall t \geq t_k + \delta$. ■

The existence of a unique $\gamma$ can be assured if $(R_{11} - R_{12}R_{22}^{-1}R_{21})$ is invertible. Using Schur complement condition for positive definiteness, it can be shown that $(R_{11} - R_{12}R_{22}^{-1}R_{21})$ is invertible iff $R(t_k + \delta, t_k) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$

is invertible. From the expressions of $R_{ij}$, it can be written that $R(t_k+\delta, t_k) = \int_{t_k}^{t_k+\delta} \mathbb{S}(t)dt$, where $\mathbb{S}(t) = \begin{bmatrix} S_1(t)S_1(t)^T & S_1(t)S_2(t)^T \\ S_2(t)S_1(t)^T & S_2(t)S_2(t)^T \end{bmatrix}$ is a positive semidefinite matrix. Therefore, the rank of $R$ is a nondecreasing function of $\delta$.

**Theorem III.2.** *If $\exists t_k' > t_k$ such that the smallest singular value of the matrix $\Phi_{\tilde{A}}(t_k, t_k') - \Phi_A(t_k, t_k')$ is strictly positive, then the $\delta$ in Theorem III.1 is $t_k' - t_k$.*

*Proof:* To guarantee the existence and uniqueness of $\gamma$, $\delta$ has to be such that $R(t_k+\delta, t_k) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}$ is invertible. Let $\tau$ be such that $t_k+\tau \geq t_k'$ and $R(t_k+\tau, t_k)$ is singular. Therefore, $\exists q \in \mathbb{R}^{2n}$ such that $q^T R(t_k, \tau, t_k)q = 0$. Hence, $\int_{t_k}^{t_k+\tau} q^T \mathbb{S}(t)q = 0$. Letting $q = [q_1^T,\ q_2^T]^T$, where $q_1,\ q_2 \in \mathbb{R}^n$, $\int_{t_k}^{t_k+\tau} \|S_1(t)^T q_1 + S_2(t)^T q_2\|_2^2 dt = 0$ is obtained. This requires for all $t \in [t_k, t_k+\tau]$,

$$S_1(t)^T q_1 + S_2(t)^T q_2 \equiv 0 \tag{20}$$

Since $S_1(t_k) = \mathbf{0}_{n\times n}$ and $S_2(t_k) = \mathbb{I}_{n\times n}$, evaluating (20) at $t_k$ gives $q_2 = 0$. Therefore, for all $t \in [t_k,\ t_k+\tau]$, $S_1(t)^T q_1 = 0$. Since $S_1(t) = \Phi_A(t_k, t) - \Phi_{\tilde{A}}(t_k, t)$, by the hypothesis of this theorem, $S_1(t_k')^T q_1 \neq 0$, which is a contradiction. Thus, $R(t_k+\tau,\ t_k)$ is nonsingular.

Since $R(t_k+\tau,\ t_k)$ is positive definite for all $\tau \geq t_k' - t_k$, $R(t_k',\ t_k)$ is positive definite and hence $\delta = t_k' - t_k$ ■

**Theorem III.3.** *For a time invariant system, $\delta$ in Theorem III.1 can be made arbitrarily small iff $(A, BK)$ is a controllable pair.*

*Proof:* From Theorem III.1, it is clear that once $R(t_k+\delta, t_k)$ has full rank, it can be ensured that for any finite value of $e(t_k)$, a unique $\gamma$ always exits. This theorem aims to prove that $\delta$ can be arbitrarily small if $(A,\ BK)$ is controllable. Let us assume $\delta$ can not be arbitrarily small and $d > 0$ be the duration such that $R(t_k+d, t_k)$ does not have full rank.

Proceeding in a way similar to Theorem III.2 leads to the fact that for some $q_1(\neq 0) \in \mathbb{R}^n$, $q_1^T S_1(t) = 0\ \ \forall t \in [t_k, t_k+d]$. It will be shown that $q_1$ has to be zero and that will prove the claim of this theorem. It is clear from the expression of $S_1(t)$ that it satisfies the differential equation (21).

$$\dot{S_1}(t) = -(S_1)A - \Phi_{\tilde{A}}(t_k, t)BK \tag{21}$$

Since $q_1^T S_1(t) \equiv 0$ for all $t \in [t_k, t_k+d]$, all the derivatives of $q_1^T S_1(t)$ must be 0 as well in that interval. Therefore using (21), it can be written that $q_1^T \dot{S_1}(t) = -q_1^T \Phi_{\tilde{A}}(t_k, t)BK$. The $m$-th derivative of $q_1^T S_1(t)$ is $q_1^T (S_1(t))^{(m)} = (-1)^m q_1^T \Phi_{\tilde{A}}(t_k, t)\tilde{A}^{m-1}BK$. Evaluating all these derivatives (only $n$ of them are needed because using Cayley-Hamilton theorem $\tilde{A}^n$ can be represented as linear combination of $\tilde{A}^i$, $i = 1, 2, ...n-1$. $n$ is the dimension of the state- space) at $t_k$ and equating them to 0 leads to $q_1^T \tilde{A}^{m-1}BK \equiv 0$ for all $m = 1, \cdots n$. With simple calculations it can be shown that $q_1^T A^{m-1}BK \equiv 0$ for all $m = 1, \cdots n$.
Therefore $q_1 \in \cap_{m=1}^n Kernel((A^{m-1}BK)^T) = \cap_{m=1}^n [Range(A^{m-1}BK)]^\perp = [\cup_{m=1}^n Range(A^{m-1}BK)]^\perp$. Since $(A, BK)$ is a controllable pair, $\cup_{m=1}^n Range(A^{m-1}BK) = \mathbb{R}^n$ and consequently $[\cup_{m=1}^n Range(A^{m-1}BK)]^\perp = 0$ leading to the fact that $q_1 = 0$, which is the desired contradiction.

For the only if part, $R(t, t_k)$ is invertible for all $t > t_k$. Therefore, for all $q \in \mathbb{R}^{2n} \setminus \{0\}$, $q^T R(t, t_k), q > 0$. Furthermore, let $q = [q_1^T, q_2^T]^T$ where $q_1 \in \cap_{m=1}^n Kernel((A^{m-1}BK)^T)$ and $q_2 = 0$.

Thus $L(t) \triangleq \int_{t_k}^t q_1^T S_1(s)S_1(s)^T q_1 ds > 0$ for all $t > t_k$. $L(t_k) = 0$ and that makes it imperative that at least one of the derivative of $L(t)$ at $t_k$ ($L^{(m)}(t_k)$) should be positive. With simple calculations it can be noticed that for $q_1 \in \cap_{m=1}^n Kernel((A^{m-1}BK)^T)$, $L^{(m)}(t_k) = 0$ for all $m$. This contradicts the fact that $L(t) > 0$ for all $t > t_k$ unless $q_1 = 0$. Hence $\cap_{m=1}^n Kernel((A^{m-1}BK)^T) = \{0\}$ and equivalently $(A,\ BK)$ is a controllable pair. ■

Note that we only considered the controllability assumption on $(A, BK)$ and this does not necessarily mean that $\tilde{A}$ is Hurwitz or the dynamics of the system (1) is stable.

## IV. Event Triggering Strategy and Optimal Noise Estimation

### A. Event Generating Function

Since $\hat{u}$ is used to drive the system (1), the system will fluctuate from its expected behavior. An event triggering strategy is implemented so that the system determines when the exact state $x(t)$ has to be transmitted to the control input generator and the behavior of the system does not go beyond the tolerance level. The goal is to keep $e(t)$ within the given tolerance level. It should be noted at this point that system (1) is run with control $\hat{u}$ and there is no reference $x_c$ available, hence $e$ is not usually known to the event generator. The event generator is designed in such a way that it actually calculates $e$ with arbitrary precision and makes decisions, based on $e(t)$, whether to trigger an event or not. Theorems III.1, III.2 and III.3 ensure that if $\psi_k(t)$ is chosen properly, the effect of the error, $e(t_k)$, on the future time ($t > t_k+\delta$) can be nullified. In fact for a time invariant controllable system, it can be done in arbitrary small time. Let us define for all $t > t_k$,

$$y(t) = \int_{t_k}^t \Phi_{\tilde{A}}(t, s)BK\xi(s)ds \tag{22}$$

From the equation of $e(t)$ in (15), if the error at $t_k$ is known and the $\psi_k(t)$, given in Theorem III.1, is chosen to update the dynamics (8), the error can be calculated for all $t \geq t_k$ by keeping track of $y(t)$. The event generator implements (22) and makes decision based on the value of $y(t)$. The $(k+1)$-th event is generated when

$$\|y(t)\| = \epsilon \tag{23}$$

where $\epsilon$ is a given tolerance level, $t_{k+1}$ denotes the time when $(k+1)$-th event is triggered. The event generator will implement a copy of the dynamics (8) so that $x_d(t)$ and consequently $x_e(t)$ are available at the event generator. The event generator depends on $\xi(t)$; and $\xi(t)$ can be calculated using (14) since $x_e(t)$ and $\psi_k(t)$ are known to the event generator. With $\psi_k(t) = 0$, the dynamics of $x_e(t)$ and $\xi(t)$ are the same and in the literature, there are several [17], [21] event triggering schemes that make decisions based on the value of $\xi(t)$. The noise being random (even for bounded amplitude noise also) causes $\xi(t)$ to fluctuate excessively, but the integral in (22) mitigates some of these fluctuations and hence results in a smaller number of communications. From theorem (III.3), it is evident that for a time invariant system, $\Omega(x_c) = \{x| \ \|x - x_c\| \le \epsilon\}$ is an invariant subspace.

**Theorem IV.1.** *For a time invariant system with $(A, BK)$ as a controllable pair, the error $\|e(t)\| = \|x(t) - x_c(t)\|$ is always bounded from above by $\epsilon$ if events are triggered according to (23).*

*Proof:* From theorem III.1, $e(t) = y(t)$ for all $t \ge t_k + \delta$. This $\delta$ can be made arbitrarily small (Theorem III.3) for a time invariant system. Therefore, $e(t) = y(t)$ for all $t > t_k$. The event triggering mechanism in (23) ensures that $\|e(t)\| < \epsilon$ for all $t > t_k$. ■

**Theorem IV.2.** *For a system with event triggering mechanism as given in (23), the time between two successive triggerings is bounded from below.*

*Proof:* Let the $k$-th event be triggered at time $t_k$ and $t_k+T$ be the time such that $y(t_k+T) = \epsilon$. Therefore the time between the $k$-th and $k+1$-th event is $T$.

Using the expression for $y(t_k + T)$, it can be written
$\epsilon = \|y(t_k + T)\| = \| \int_{t_k}^{t_k+T} \Phi_{\tilde{A}}(t_k + T, s)BK\xi(s)ds\|$.
$\int_{t_k}^{t_k+T} \|\Phi_{\tilde{A}}(t_k + T, s)BK\xi(s)\|ds \ge \epsilon$.
$\int_{t_k}^{t_k+T} \|\Phi_{\tilde{A}}(t_k + T, s)BK\|\|\xi(s)\|ds \ge \epsilon$.

The process $\xi(s)$ has the property that $\xi(t_k^+) = 0$ and using the Kolmogorov Continuity Theorem [23], it can be easily show that the process $\xi(s)$ has a continuous sample path probability almost surely. Therefore, $\mathcal{F}(s) \triangleq \Phi_{\tilde{A}}(t_k + T, s)\tilde{B}(s)K(s)\xi(s)$ is also continuous almost surely and its value at $s = t_k$ is zero. $\mathcal{F}(s)$ being a continuous function almost surely in a compact domain $[t_k, t_k + T]$ will have finite value almost surely and hence $\exists T_{\min} > 0$ such that for all $T < T_{\min}$, $\int_{t_k}^{t_k+T} \|\mathcal{F}(s)\|ds < \epsilon$. Therefore the interval between two successive triggerings is bounded from below by $T_{\min}$. ■

Theorem IV.2 ensures that infinite triggerings are not generated within a finite amount of time, i.e. the event generator does not exhibit Zeno behavior [24].

**Theorem IV.3.** *For a time invariant system, with event triggering mechanism governed by (23), the expected duration between triggerings at $t_k$ and $t_{k+1}$ does not depend on the history $\{t_i\}_{i=1}^k$.*

*Proof:* Let $\tau_k$ be the expected amount of time between events at $t_k$ and $t_{k+1}$. Therefore,

$\mathbb{E}\left[| \int_{t_k}^{t_k+\tau_k} \Phi_{\tilde{A}}(t_k + \tau_k, s)BK\xi(s)ds|\right] = \epsilon$.
$\mathbb{E}\left[| \int_0^{\tau_k} \Phi_{\tilde{A}}(t_k + \tau_k, t_k + r)BK\xi(t_k + r)dr|\right] = \epsilon$.
$\mathbb{E}\left[| \int_0^{\tau_k} \Phi_{\tilde{A}}(\tau_k, r)BK\xi(t_k + r)dr|\right] = \epsilon$.
$\xi(t_k + r) = \int_{t_k}^{t_k+r} \Phi_A(t_k + r, s)dW_s$ is a Gaussian random variable with zero mean and variance $\Sigma(t_k + r) = \int_{t_k}^{t_k+r} \Phi_A(t_k + r, s)\Phi_A^T(t_k + r, s)ds = \int_0^r \Phi_A(r, s)\Phi_A^T(r, s)ds = \Sigma(r)$. Since a Gaussian variable is fully characterized by its mean and variance, the random variables $\xi(t_k + r)$ and $\xi(r)$ have the same statistical properties. Hence $\mathbb{E}\left[| \int_0^{\tau_k} \Phi_{\tilde{A}}(\tau_k, r)BK\xi(t_k + r)dr|\right] = \mathbb{E}\left[| \int_0^{\tau_k} \Phi_{\tilde{A}}(\tau_k, r)BK\xi(r)dr|\right] = \epsilon$.
Clearly $\tau_k$ does not depend on any of the $t_i$s for $i = 1, 2, \cdots, k$. ■

Theorem IV.3 implies that if some apriori knowledge of expected inter-event times is available, which can be estimated from the history of previous triggerings, the expected time for the future triggerings and the expected numbers of triggerings that will be generated in a given time horizon can be calculated.

### B. Optimal Noise Estimate

The noise estimate has certain influence on the triggering mechanism and this section describes the optimal noise estimate and how it can be computed. In section III-A, (9) represents one way to improve the noise estimate in a recursive fashion. From (13), the solution of $\xi(t)$, for all $t > t_k$, can be written as,

$$\xi(t) = \int_{t_k}^t \Phi_A(t, s)[dW_s - \hat{W}_k ds] \tag{24}$$

Using (24) with (22), it is obtained that

$$y(t) = \int_{t_k}^t \Phi_{\tilde{A}}(t, r)BK\Big[\int_{t_k}^r \Phi_A(r, s)[dW_s - \hat{W}_k ds]\Big]dr$$

After some simplifications,

$$y(t) = \int_{t_k}^t (\Phi_A(t, s) - \Phi_{\tilde{A}}(t, s))[dW_s - \hat{W}_k ds] \tag{25}$$

If $dW_s$ is a constant value disturbance i.e. $\mathbb{E}[W_s] = \hat{w}$, and $Var(W_s) = 0$, $y(t) = \mathbb{E}[y(t)] = \int_{s=t_k}^t (\Phi_A(t, s) - \Phi_{\tilde{A}}(t, s))[\hat{w} - \hat{W}_k]ds$ which can be made zero if $\hat{W}_k = \hat{w}$. This can be achieved by the recursive estimation given in (9).

If $dW_s$ is a Wiener process - i.e. $\mathbb{E}[dW_s] = 0$ and $\mathbb{E}(dW_s dW_t^T) = \mathbf{1}_{n\times n}\delta(t - s)dt$, where $\delta(\cdot)$ is the delta function- $y(t)$ will be a Gaussian process with mean $\mu_t = -\int_{s=t_k}^t (\Phi_A(t, s) - \Phi_{\tilde{A}}(t, s))\hat{W}_k ds$ and variance $\Sigma_t = \int_{t_k}^t (\Phi_A(t, s) - \Phi_{\tilde{A}}(t, s))(\Phi_A(t, s) - \Phi_{\tilde{A}}(t, s))^T ds$.
Therefore, when $\Sigma_t$ is invertible,
$P(\|y(t)\| < \epsilon) = \int_{\|y\|<\epsilon} \frac{1}{(2\pi det(\Sigma_t))^{n/2}} e^{-\frac{1}{2}(y-\mu_t)^T \Sigma_t{}^{-1}(y-\mu_t)}dy$.
This probability will be maximized iff $\mu_t = 0$. This implies that the optimal $\hat{W}_k$ has to be zero for all $k$. The optimal $\hat{W}_k$ increases the probability that $\|y(t)\| < \epsilon$ and hence it reduces the expected number of triggerings.

**Remark IV.4.** *For a time invariant system, $\Sigma_t$ is positive definite for all $t > t_k$ iff $(\tilde{A}, BK)$ is a controllable pair.*

Remark IV.4 can be proved by following steps similar to that in Theorem III.3.

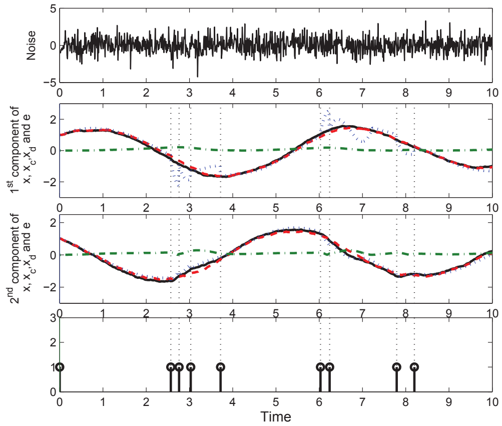


Fig. 2. Performance of the system under noise modeled by Wiener process. The first plot represents one noise component. The second and third plots represent the $1^{st}$ and $2^{nd}$ components of $x, x_c, x_d$ and $e$. The blue dotted curve represents $x_d$, the red one represents $x_c$, the black one represents $x$ and the green dashed line represents the error $e$. The fourth plot shows the time instances when the events were triggered.

## V. Simulation Results

Let us consider the dynamics (26),

$$d\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}dt + \begin{bmatrix}0\\1\end{bmatrix}udt + dW_t \qquad (26)$$

The aim is to maintain a circular phase portrait and hence $u$ is chosen to be $-x_1$. Therefore, for this system $K = [1, 0]$, $BK = \begin{bmatrix}0\\1\end{bmatrix}[1,0] = \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}$, $\tilde{A} = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$. The initial condition for this system is $[1,\ 1]^T$. Clearly $(A, BK)$ is a controllable pair. The matrix $\tilde{A}$ is not Hurwitz and the closed loop system is not asymptotically stable (it has poles on imaginary axis).

In the first experiment, a Wiener process noise is applied and the performance of the system is shown in Figure 2. The norm in (23) is chosen to be the supremum norm i.e. $\|y(t)\| = \max\{|y_1(t)|,\ |y_2(t)|\}$ and $\epsilon = 0.2$. The blue dotted curves in the second and third plots in Figure 2 are the components of $x_d(t)$. Since at each event triggering the dynamics of $x_d$ are updated by introducing $\psi_k(t)$, the dynamics of $x_d$ show some large deviation from $x$. The phase portrait of the system is given in Figure 3, where the blue curve is for $x$, the red one for $x_c$ and the black dashed curve for $x_d$ , and the blue and red circles represent the time instances the events were triggered. As seen from Figure 3, at the triggering instances, $x_d$ updates its dynamics with proper $\psi_k(t)$ (obtained using Theorem III.1) and that reduces the error between the red and blue curve. As found in section IV-B, the noise estimate used is 0. The system does not show Zeno behavior and 8 triggerings are needed in the time interval $[0, 10]$.

In the second experiment a constant noise signal is considered. The noise and the performance of the system is plotted in Figure 4. Since noise is constant, only one triggering is sufficient to determine the noise value. After the triggering, the $x$ and $x_d$ processes follow the same trajectory. The phase portrait in Figure 5 shows that there was some error due to control mismatch but the error diminishes gradually due to the control $\psi_1(t)$.

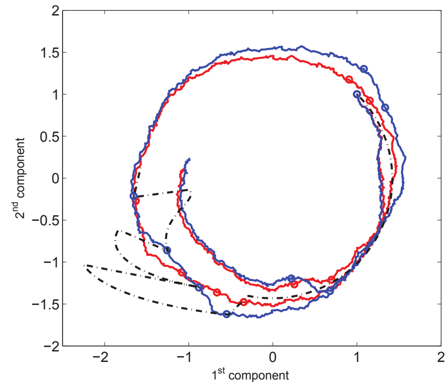


Fig. 3. Phase portrait for $x$, $x_d$ and $x_c$. The blue curve represents $x$, while the red and dashed black ones represent the $x_c$ and $x_d$ respectively. To maintain some clarity $x_d$ is plotted for some initial time only. All the trajectories start at the point (1,1).

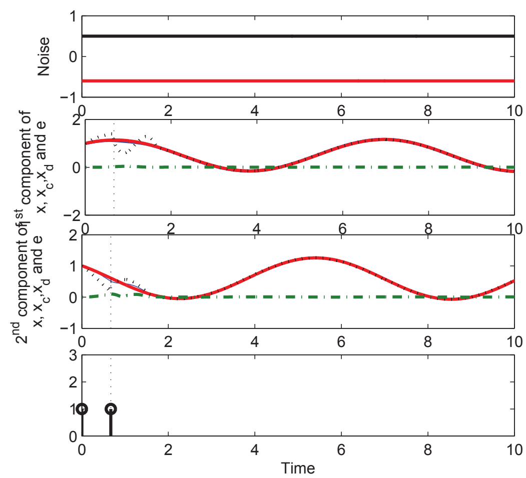


Fig. 4. Performance of the system under constant noise. The first plot represents both components of the noise. The second and third plots represent the $1^{st}$ and $2^{nd}$ components of $x, x_c, x_d$ and $e$. The blue curve represents $x$, the red one represents $x_c$, the black one represents $x_d$ and the green dashed line represents the error $e$. The fourth plot shows the time instances when the events were triggered.

In the same set up another experiment was performed where $\psi_k(t) = 0$ was used. The phase portrait for this case is shown in Figure 6 where it can be seen that the error between the blue and red curves never decreases. This ensures that $\psi_k(t)$ is equally important even in the case of a constant noise. The error is periodic due to the structure of the $\tilde{A}$ matrix.

In the next experiment, a Wiener process noise is considered again but two different approaches were used to estimate the noise. From Section IV-B, the optimal estimate $\hat{W}_k$ is zero for all $k$, and (9) also provides an estimate of the same. With $\hat{W}_k = 0$, only five triggerings (excluding the one at time 0) are required whereas using (9) requires eight triggerings. This is shown in Figure 7. The triggering at time 0 sets the initial values for the states of systems.

In summary, it is demonstrated that $\psi_k(t)$ is equally

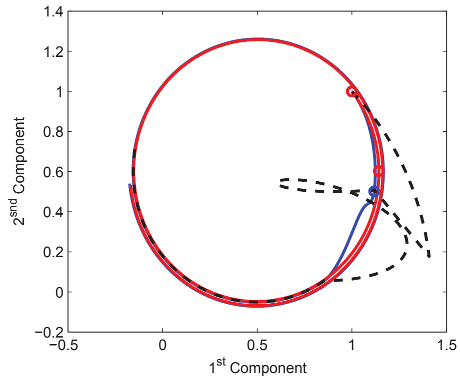


Fig. 5. Phase portrait for $x$, $x_d$ and $x_c$. The blue curve represents $x$, while the red and dashed black ones represent the $x_c$ and $x_d$ respectively. To maintain some clarity $x_d$ is plotted for some initial time only. All the trajectories start at the point (1,1).

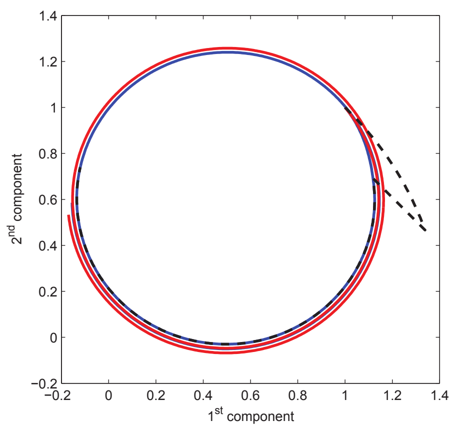


Fig. 6. Phase portrait for $x$, $x_d$ and $x_c$. The blue curve represents $x$, while the red and dashed black ones represent the $x_c$ and $x_d$ respectively. To maintain some clarity $x_d$ is plotted for some initial time only. All the trajectories start at the point (1,1). There is a constant error between $x(t)$ and $x_c(t)$ and it never decreases.

important to make the event based system mimic the closed loop system and results from Theorem III.1 is used to obtain $\psi_k(t)$. From the simulation results, it is clear that the error between event based system and closed loop system is bounded as stated in Theorem IV.1.

## VI. Conclusion

In this paper, a controller and an event triggering mechanism is proposed for an event based control scheme. Despite of the lack of continuous state feedback, the error $e$ incurred by this event based scheme can still be calculated accurately. The event triggering scheme triggers an event based on this error. The controller updates the system in such a way that decreases the error accumulated due to control input mismatch. Theorem III.1 provides an explicit formula for the control that minimizes this error. Theorem III.2 gives an indication how fast this error can be minimized. Theorem IV.2 proves that there is minimum time between two triggerings and hence the triggering mechanism does not exhibit Zeno behavior. These results are true for any linear system (stable or unstable) expect for a few features that are only applicable for time invariant systems

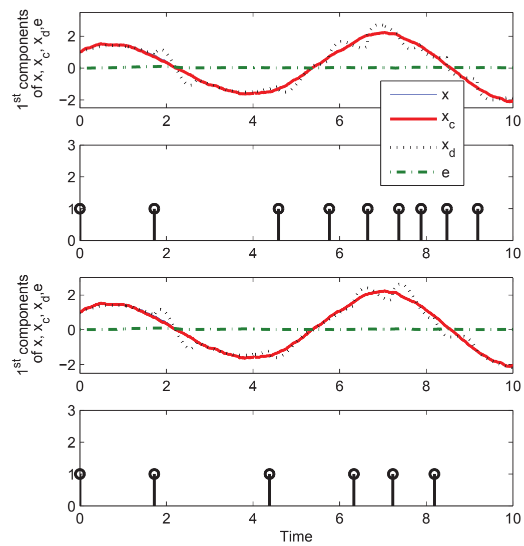


Fig. 7. Performance of the system under same Wiener process noise but with different noise estimate. The first plot represents the $1^{st}$ component of $x, x_c, x_d$ and $e$ and the second plot is the corresponding event triggering instances with optimal noise estimate to be zero. The third and fourth plots are similar to first and second respectively but with the noise estimate given in (9).

with certain controllability assumption (Theorems III.3, IV.1, IV.3). The experimental results justify the theoretical findings.

A possible future work would be to extend this analysis for non-linear systems or systems with delays and dropouts.

## Acknowledgment

This work is partially supported by US Air Force Office of Scientific Research MURI grant FA9550-09-1-0538, by DARPA through ARO grant W911NF1410384, by National Science Foundation (NSF) grant CNS- 1035655, and by National Institute of Standards and Technology (NIST) grant 70NANB11H148.

## References

[1] K. J. Åström and B. Bernhardsson, "Comparison of Riemann and Lebesque sampling for first order stochastic systems," in *Proceedings of the 41st IEEE Conference on Decision and Control, 2002*, vol. 2. IEEE, 2002, pp. 2011–2016.

[2] M. Velasco, J. Fuertes, and P. Marti, "The self triggered task model for real-time control systems," in *Work-in-Progress Session of the 24th IEEE Real-Time Systems Symposium (RTSS03)*, vol. 384, 2003.

[3] M. Lemmon, T. Chantem, X. S. Hu, and M. Zyskowski, "On self-triggered full-information H-infinity controllers," in *Hybrid Systems: computation and control*. Springer, 2007, pp. 371–384.

[4] D. Nešić, A. R. Teel, and P. Kokotović, "Sufficient conditions for stabilization of sampled-data nonlinear systems via discrete-time approximations," *Systems & Control Letters*, vol. 38, no. 4, pp. 259–270, 1999.

[5] W. Bian and M. French, "General fast sampling theorems for nonlinear systems," *Systems & control letters*, vol. 54, no. 11, pp. 1037–1050, 2005.

[6] Y. Zheng, D. Owens, and S. Billings, "Fast sampling and stability of nonlinear sampled-data systems: Part 2. sampling rate estimations," *IMA Journal of Mathematical Control and Information*, vol. 7, no. 1, pp. 13–33, 1990.

[7] K. J. Åström and B. Bernhardsson, "Comparison of periodic and event based sampling for first-order stochastic systems," in *Proceedings of the 14th IFAC World congress*, vol. 11. Citeseer, 1999, pp. 301–306.

[8] K.-E. Årzén, "A simple event-based PID controller," in *Proc. 14th IFAC World Congress*, vol. 18, 1999, pp. 423–428.

[9] W. Heemels, R. Gorter, A. van Zijl, P. Van den Bosch, S. Weiland, W. Hendrix, and M. Vonder, "Asynchronous measurement and control: a case study on motor synchronization," *Control Engineering Practice*, vol. 7, no. 12, pp. 1467–1482, 1999.

[10] K. J. Aström, "Event based control," in *Analysis and design of nonlinear control systems*. Springer, 2008, pp. 127–147.

[11] W. Heemels, J. Sandee, and P. Van Den Bosch, "Analysis of event-driven controllers for linear systems," *International journal of control*, vol. 81, no. 4, pp. 571–590, 2008.

[12] P. G. Otanez, J. R. Moyne, and D. M. Tilbury, "Using deadbands to reduce communication in networked control systems," in *American Control Conference, 2002. Proceedings of the 2002*, vol. 4. IEEE, 2002, pp. 3015–3020.

[13] T. Henningsson, E. Johannesson, and A. Cervin, "Sporadic event-based control of first-order linear stochastic systems," *Automatica*, vol. 44, no. 11, pp. 2890–2895, 2008.

[14] C. De Persis, "Robust stabilization of nonlinear systems by quantized and ternary control," *Systems & Control Letters*, vol. 58, no. 8, pp. 602–608, 2009.

[15] C. De Persis and A. Isidori, "Stabilizability by state feedback implies stabilizability by encoded state feedback," *Systems & Control Letters*, vol. 53, no. 3, pp. 249–258, 2004.

[16] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints. II. Stabilization with limited information feedback," *Automatic Control, IEEE Transactions on*, vol. 44, no. 5, pp. 1049–1053, 1999.

[17] J. Lunze and D. Lehmann, "A state-feedback approach to event-based control," *Automatica*, vol. 46, no. 1, pp. 211–215, 2010.

[18] D. V. Dimarogonas and K. H. Johansson, "Event-triggered control for multi-agent systems," in *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*. IEEE, 2009, pp. 7131–7136.

[19] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, "Control of multi-agent systems via event-based communication," in *Proceedings of the 18th IFAC World Congress, 2011*, 2011.

[20] R. Cogill, "Event-based control using quadratic approximate value functions," in *Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on*. IEEE, 2009, pp. 5883–5888.

[21] M. Guinaldo, D. V. Dimarogonas, K. H. Johansson, J. Sánchez, and S. Dormido, "Distributed event-based control for interconnected linear systems," in *Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on*. IEEE, 2011, pp. 2553–2558.

[22] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, "Event-based broadcasting for multi-agent average consensus," *Automatica*, vol. 49, no. 1, pp. 245–252, 2013.

[23] I. Karatzas, *Brownian motion and stochastic calculus*. Springer Science & Business Media, 1991, vol. 113.

[24] A. D. Ames, A. Abate, and S. Sastry, "Sufficient conditions for the existence of Zeno behavior," in *Decision and Control, 2005 and 2005 European Control Conference. CDC-ECC'05. 44th IEEE Conference on*. IEEE, 2005, pp. 696–701.